Integral formula and upper estimate of I and J-Bessel functions on Jordan algebras

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Abstract

In this paper we give a new integral expression of I and J-Bessel functions on simple Euclidean Jordan algebras, integrating on a bounded symmetric domain. From this we easily get the upper estimate of Bessel functions. As an application we state about the action of holomorphic 1-dimensional semi-group acting on the space of square integrable functions on symmetric cones.

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1 Introduction and main results

In this paper we find in Theorem 3.1 a new integral expression of I and J-Bessel functions $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ on a Jordan algebra V. J-Bessel functions are first introduced by Faraut and Travaglini [8] for special cases, associating to self-adjoint representations of Jordan algebras (see also (4.2)), and generalized by Dib [4] (for $V = \mathrm{Sym}(r,\mathbb{R})$ case see also [10] and [15]). It is well-known that $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ are the holomorphic functions on $V^{\mathbb{C}}$ for λ in open dense subset of $\mathbb{C}$. On the other hand, for countable singular λ they are still well-defined on certain subvarieties. These are defined by the series expansion (see Section 3), and satisfy the following differential equation

$$\mathcal{B}_\lambda\mathcal{I}_\lambda - e\mathcal{I}_\lambda = 0, \qquad \mathcal{B}_\lambda\mathcal{J}_\lambda + e\mathcal{J}_\lambda = 0$$

where $\mathcal{B}_\lambda : C^2(V) \to C(V)\otimes V^{\mathbb{C}}$ is the $V^{\mathbb{C}}$-valued 2nd order differential operator and e is the unit element on V (see [4, Proposition 1.7] or [6, Theorem XV.2.6]). Also $\mathcal{I}_\lambda$ and $\mathcal{J}_\lambda$ have the following integral expression

$$\mathcal{I}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n}\int_{e+iV} e^{\operatorname{tr} w} e^{(w^{-1}|x)}\Delta(w)^{-\lambda}dw, \tag{1.1}$$

$$\mathcal{J}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n}\int_{e+iV} e^{\operatorname{tr} w} e^{-(w^{-1}|x)}\Delta(w)^{-\lambda}dw \tag{1.2}$$

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(see [4, Définition 1.2] or [6, Theorem XV.2.2]. For notations tr, $(\cdot|\cdot)$, Δ and $\Gamma_\Omega(\lambda)$ see Section 2.1 and (2.3)). We briefly state our theorem.

Let V be a simple Euclidean Jordan algebra (*i.e.*, V is one of the $\mathrm{Sym}(r,\mathbb{R})$, $\mathrm{Herm}(r,\mathbb{C})$, $\mathrm{Herm}(r,\mathbb{H})$, $\mathbb{R}^{1,n-1}$ or $\mathrm{Herm}(3,\mathbb{O})$). We assume $\dim V = n$, $\operatorname{rank} V = r$. We prove

Theorem 1.1. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$ (see (2.1) and (2.6)), take $k \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$. Then, we have the integral expressions*

$$\mathcal{I}_\lambda\left(x^2\right) = c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-x,w)e^{2(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw,$$
$$\mathcal{J}_\lambda\left(x^2\right) = c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-ix,w)e^{2i(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw.$$

where c_λ is a constant and ${}_1F_1(-k,\lambda;x,w)$ is a polynomial of degree k with respect to both x and w.

Here $\mathcal{X}_l$ are the $L = \mathrm{Str}(V^{\mathbb{C}})_0$-orbits. $\overline{\mathcal{X}_l}$ are also characterized as the supports of some distributions on $V^{\mathbb{C}}$ (see [2] and (2.2)). $D \subset V^{\mathbb{C}}$ is the *bounded symmetric domain* and $h(w,w)$ is the *generic norm* on $V^{\mathbb{C}}$ (see Section 2.1). Especially if $\operatorname{Re}\lambda > \frac{2n}{r} - 1$ we can take $k=0$ and

$$\mathcal{I}_\lambda\left(x^2\right) = \frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda-\frac{n}{r}\right)}\int_D e^{2(x|\operatorname{Re}w)}h(w,w)^{\lambda-\frac{2n}{r}}dw$$

and $\mathcal{J}_\lambda$ is the same.

Now D is naturally identified with $G/K = \mathrm{Bihol}(D)/\mathrm{Stab}(0) = \mathrm{Co}(V)_0/\mathrm{Aut}_{\mathrm{JTS}}(V)_0$. For $\lambda > \frac{2n}{r} - 1$, the universal covering group $\tilde{G}$ acts unitarily on $\mathcal{O}(D) \cap L^2(D, h(w,w)^{\lambda-\frac{2n}{r}}dw)$ by left translation. This defines the holomorphic discrete series representation of $\tilde{G}$. This is analytically continued with respect to $\lambda \in \mathbb{C}$, and become unitary when $\lambda \in \mathcal{W}$, the *(Berezin–)Wallach set* (see (2.7) and [17], [3]). The trivial representation corresponds to $\lambda = 0$.

From now we set $V = \mathbb{R}$. Let $I_\lambda(x)$ be the classical I-Bessel function and we set $\tilde{I}_\lambda(x) = \left(\frac{x}{2}\right)^{-\lambda} I_\lambda(x)$. Then $\tilde{I}_\lambda$ and $\mathcal{I}_\lambda$ on $\mathbb{R}$ are related as

$$\tilde{I}_\lambda(x) = \frac{1}{\Gamma(\lambda+1)}\mathcal{I}_{\lambda+1}\left(\frac{x^2}{4}\right).$$

Therefore the above theorem is rewritten as

$$\tilde{I}_\lambda(x) = \frac{\lambda+k}{\pi\Gamma(\lambda+1)}\int_{|w|<1} {}_1F_1(-k,\lambda+1;-xw)e^{x\operatorname{Re}w}\left(1-|w|^2\right)^{\lambda+k-1}dw.$$

where ${}_1F_1(-k,\lambda+1;x)$ is the classical hypergeometric polynomial. On the other hand, the formula (1.1) is rewritten as

$$\tilde{I}_\lambda(x) = \frac{1}{2i\pi\lambda}\int_{1+i\mathbb{R}} e^{w+\frac{x^2}{w}}w^{-\lambda-1}dw.$$

These two integral formulas are mutually independent, and cannot easily deduce one from another.

Again let V be a general Jordan algebra. Since D is bounded, we can prove from this formula the following corollary.

Corollary 1.2. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, if $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k} > 0$ such that*

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k} \left(1 + |x|_1^k\right) e^{2|\operatorname{Re} x|_1}, \qquad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k} \left(1 + |x|_1^k\right) e^{2|\operatorname{Im} x|_1}$$

where $|x|_1$ is the norm defined in Definition 2.1.

In [14, Lemma 3.1] an upper estimate of $\mathcal{J}_\lambda(x)$ is given by another method, but our estimate is sharper. For detail see Remark 3.3.

This paper is organized as follows: In Section 2, we recall some notations and facts about Euclidean Jordan algebras. In Section 3 we prove our main theorem, the integral formula and upper estimates. In Section 4, as an application of the inequality (Corollary 1.2), we state the action of the 1-dimensional semigroup on the functions on the symmetric cones.

2 Preliminaries

2.1 Simple Euclidean Jordan algebras

Let V be a simple Euclidean Jordan algebra of dimension n, rank r. We denote the unit element by e. Also let $V^{\mathbb{C}}$ be its complexification. For $x, y, z \in V^{\mathbb{C}}$, we write

$$\begin{aligned} L(x)y &:= xy, \\ x\Box y &:= L(xy) + [L(x), L(y)], \\ P(x,z) &:= L(x)L(z) + L(z)L(x) - L(xz), \\ P(x) &:= P(x,x) = 2L(x)^2 - L(x^2), \\ B(x,y) &:= I_{V^{\mathbb{C}}} - 2x\Box\bar{y} + P(x)P(\bar{y}) \end{aligned}$$

where $y \mapsto \bar{y}$ is the complex conjugation with respect to the real form V. Also, we write

$$\{x,y,z\} := (x\Box\bar{y})z = P(x,z)\bar{y} = (x\bar{y})z + x(\bar{y}z) - (xz)\bar{y}.$$

Then $V^{\mathbb{C}}$ becomes a Hermitian positive Jordan triple system with this triple product.

We denote the *Jordan trace* and the *Jordan determinant* of the complex Jordan algebra $V^{\mathbb{C}}$ by $\operatorname{tr}(x)$ and $\Delta(x)$ respectively. Also let $h(x,y)$ be the *generic norm* of the Jordan triple system $V^{\mathbb{C}}$. These can be expressed by $L(x)$, $P(x)$, and $B(x,y)$ (see [6, Proposition III.4.2], [7, Part V, Proposition VI.3.6]):

$$\begin{aligned} \operatorname{Tr} L(x) &= \frac{n}{r}\operatorname{tr}(x), \\ \operatorname{Det} P(x) &= \Delta(x)^{\frac{2n}{r}}, \\ \operatorname{Det} B(x,y) &= h(x,y)^{\frac{2n}{r}} \end{aligned}$$

where Tr and Det stand for the usual trace and determinant of complex linear operators on $V^{\mathbb{C}}$. Using the Jordan trace we define the inner product on $V^{\mathbb{C}}$:

$$(x|y) := \operatorname{tr}(x\bar{y}), \qquad x, y \in V^{\mathbb{C}}.$$

Then this is positive definite since V is Euclidean. Also we define the *symmetric cone* Ω and the *bounded symmetric domain* D by

$$\Omega := \{x^2 : x \in V,\ \Delta(x) \neq 0\},$$
$$D := (\text{connected component of } \{w \in V^{\mathbb{C}} : h(w,w) \gg 0\} \text{ which contains } 0)$$

where $\gg$ stands for the positive definiteness. Then Ω is self-dual, *i.e.*,

$$\Omega = \{x \in V : (x|y) > 0\ for\ \forall y \in \Omega\},$$

and D is biholomorphically equivalent to $V + \sqrt{-1}\Omega \subset V^{\mathbb{C}}$.

Let K_L and K be the identity components of *automorphism groups* of the Jordan algebra V and the Jordan triple system $V^{\mathbb{C}}$. Similarly let L and $L^{\mathbb{C}}$ be the identity components of *structure groups* of V and $V^{\mathbb{C}}$. Also let G be the identity component of *conformal group* of V:

$$K_L := \mathrm{Aut}_{\mathrm{J.Alg}}(V)_0 = \{k \in GL(V) : k(xy) = kx \cdot ky,\ \forall x, y \in V\}_0,$$
$$K := \mathrm{Aut}_{\mathrm{JTS}}(V^{\mathbb{C}})_0 = \{k \in GL(V^{\mathbb{C}}) : k\{x,y,z\} = \{kx, ky, kz\},\ \forall x, y, z \in V^{\mathbb{C}}\}_0,$$
$$L := \mathrm{Str}(V)_0 = \{l \in GL(V) : l\{x,y,z\} = \{lx, {}^t l^{-1} y, kz\},\ \forall x, y, z \in V\}_0,$$
$$L^{\mathbb{C}} := \mathrm{Str}(V^{\mathbb{C}})_0 = \{l \in GL(V^{\mathbb{C}}) : l\{x,y,z\} = \{lx, l^{*-1} y, kz\},\ \forall x, y, z \in V^{\mathbb{C}}\}_0,$$
$$G := \mathrm{Co}(V)_0 = \mathrm{Bihol}(D)_0 \simeq \mathrm{Bihol}(V + \sqrt{-1}\Omega)_0$$

Then Ω and D are naturally identified with L/K_L and G/K respectively. For the classification of these groups see [11, Table 1] or [14, Table 1].

2.2 Spectral decomposition and some norms on $V^{\mathbb{C}}$

From now on we fix a *Jordan frame* $\{c_1, \ldots, c_r\} \subset V$, *i.e.*,

$$c_j c_k = \delta_{jk} c_j, \qquad \sum_{j=1}^{r} c_j = e,$$
$$\textit{and there do not exist } d_{j1}, d_{j2} \in V \textit{ s.t. } c_j = d_{j1} + d_{j2},\ d_{jk} d_{jl} = \delta_{kl} d_{jk}.$$

Then for any $x \in V^{\mathbb{C}}$ there exist the unique numbers $t_1 \geq \cdots t_r \geq 0$ and the element $k \in K$ such that $x = k \sum_{j=1}^{r} t_j c_j$ ([6, Proposition X.3.2]). Using this, we define the p-norm on $V^{\mathbb{C}}$.

Definition 2.1. *For* $1 \leq p \leq \infty$ *and for* $x = k \sum_{j=1}^{r} t_j c_j \in V^{\mathbb{C}}$*, we define*

$$|x|_p := \begin{cases} \left(\displaystyle\sum_{j=1}^{r} |t_j|^p \right)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \displaystyle\max_{j \in \{1, \ldots, r\}} |t_j| & (p = \infty). \end{cases}$$

For example, we have $(x|x) = |x|_2^2$. Also if $x \in \Omega$ then all eigenvalues (in the sense of Jordan algebras. For $V = \mathrm{Sym}(r, \mathbb{R})$ or $\mathrm{Herm}(r, \mathbb{C})$ this coincides with the usual one) are positive and $|x|_1 = \mathrm{tr}\, x$ holds. In addition, we can define D by $D = \{w \in V^{\mathbb{C}} : |w|_\infty < 1\}$. This norm satisfies the following properties.

Proposition 2.2 ([16, Theorem V.4, V.5] for $V = \mathrm{Herm}(r, \mathbb{C})$ case)**.** *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements hold.*

(1) For $x, y \in V^{\mathbb{C}}$, $|(x|y)| \leq |x|_p |y|_q$.

(2) For $x \in V^{\mathbb{C}}$, $|x|_p = \displaystyle\max_{y \in V^{\mathbb{C}} \setminus \{0\}} \frac{|(x|y)|}{|y|_q}$.

(3) $x \mapsto |x|_p$ is a norm on $V^{\mathbb{C}}$.

To prove this, we quote the following lemma (see [7, Part V, Proposition VI.2.1]):

Lemma 2.3. *For $x, y \in V^{\mathbb{C}}$, if $x \square \bar{y} = y \square \bar{x}$, then there exists an element $k \in K$ such that both x and y belong to $\mathbb{R}\text{-}\mathrm{span}\{kc_1, \dots, kc_r\}$.*

Proof of Proposition 2.2. (1) We note that $|(x|y)| \leq \max_{k \in K} |(kx|y)| = \max_{k \in K} \mathrm{Re}(kx|y)$ since $e^{i\theta} I_{V^{\mathbb{C}}} \in K$ for any $\theta \in \mathbb{R}$. We take $k_0 \in K$ such that $\mathrm{Re}(kx|y)$ $(k \in K)$ attains its maximum at $k = k_0 \in K$. We put $k_0 x =: x_0$. Then for any $D \in \mathfrak{k} = Lie(K)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Re}(e^{tD} x_0|y) = \mathrm{Re}(Dx_0|y) = 0.$$

In the case when $D = u \square \bar{v} - v \square \bar{u}$ with $u, v \in V^{\mathbb{C}}$,

$$\begin{aligned} 0 &= \mathrm{Re}((u \square \bar{v}) x_0|y) - \mathrm{Re}((v \square \bar{u}) x_0|y) = \mathrm{Re}((x_0 \square \bar{v}) u|y) - \mathrm{Re}((x_0 \square \bar{u}) v|y) \\ &= \mathrm{Re}(u|(v \square \bar{x_0}) y) - \mathrm{Re}(v|(u \square \bar{x_0}) y) = \mathrm{Re}(u|(y \square \bar{x_0}) v) - \mathrm{Re}(v|(y \square \bar{x_0}) u) \\ &= \mathrm{Re}((x_0 \square \bar{y}) u|v) - \mathrm{Re}(v|(y \square \bar{x_0}) u) = \mathrm{Re}((x_0 \square \bar{y} - y \square \bar{x_0}) u|v). \end{aligned}$$

Since $u, v \in V^{\mathbb{C}}$ are arbitrary and $(\cdot|\cdot)$ is non-degenerate, $x_0 \square \bar{y} = y \square \bar{x_0}$. Therefore by Lemma 2.3 there exists $k \in K$ such that $x_0, y \in \mathbb{R}\text{-}\mathrm{span}\{kc_1, \dots, kc_r\}$. Let $x = k' \sum_{j=1}^r t_j c_j$, $y = k \sum_{j=1}^r s_j c_j$. Then

$$\begin{aligned} |(x|y)| &\leq \max_{k \in K} \mathrm{Re}(kx|y) = \max_{(\sigma, \varepsilon) \in \mathfrak{S}_n \ltimes \{\pm 1\}^r} \mathrm{Re} \left(k \sum_{j=1}^r \varepsilon_j t_{\sigma(j)} c_j \,\middle|\, k \sum_{j=1}^r s_j c_j \right) \\ &= \max_{(\sigma, \varepsilon) \in \mathfrak{S}_n \ltimes \{\pm 1\}^r} \sum_{j=1}^r \varepsilon_j t_{\sigma(j)} s_j \leq \left(\sum_{j=1}^r |t_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^r |s_j|^q \right)^{\frac{1}{q}} = |x|_p |y|_q. \end{aligned}$$

(2) ($\geq$) Clear from (1).
($\leq$) For $x = k \sum_{j=1}^r t_j c_j \in V^{\mathbb{C}}$ $(t_1 \geq \cdots t_r \geq 0)$, we find a $y \in V^{\mathbb{C}}$ which attains the equality. We set

$$y := \begin{cases} k \sum_{j=1}^r t_j^{p-1} c_j & (1 \leq p < \infty), \\ kc_1 & (p = \infty). \end{cases}$$

Then,

$$|y|_q = \begin{cases} \left(\sum_{j=1}^r t_j^{(p-1)q} \right)^{\frac{1}{q}} = \left(\sum_{j=1}^r t_j^{\frac{1}{p}} \right)^{\frac{p-1}{p}} = |x|_p^{p-1} & (1 < p < \infty), \\ 1 & (p = 1, \infty), \end{cases}$$

and

$$(x|y) = \begin{cases} \sum_{j=1}^{r} t_j^p = |x|_p^p = |x|_p |x|_p^{p-1} = |x|_p |y|_q & (1 \le p < \infty), \\ t_1 = |x|_\infty = |x|_\infty |y|_1 & (p = \infty). \end{cases}$$

(3) Positivity and homogeneity are clear. For triangle inequality, by (2), for $x, y \in V^{\mathbb{C}}$,

$$|x+y|_p = \max_{|z|_q=1} |(x+y|z)| \le \max_{|z|_q=1} |(x|z)| + \max_{|z|_q=1} |(y|z)| = |x|_p + |y|_p$$

and this completes the proof. □

We set

$$\mathcal{X}_l := \left\{ k \sum_{j=1}^{l} t_j c_j : k \in K,\ t_j > 0 \right\} = L^{\mathbb{C}} \cdot \sum_{j=1}^{l} e_j \subset V^{\mathbb{C}} \quad (l = 0, \ldots, r). \tag{2.1}$$

Then $\overline{\mathcal{X}_l} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_l$ holds. $\overline{\mathcal{X}_l}$ are also characterized as the supports of the distributions which are the analytic continuation of $|\Delta(x)|^{2\left(\lambda - \frac{n}{r}\right)} dx$:

$$\operatorname{supp}\left(|\Delta(x)|^{2\left(\lambda - \frac{n}{r}\right)} dx \Big|_{\lambda = l\frac{d}{2}} \right) = \overline{\mathcal{X}_l}, \qquad l = 0, 1, \ldots, r-1 \tag{2.2}$$

(see [2, Proposition 5.5]).

2.3 Peirce decomposition and generalized power function

As before we fix a Jordan frame $\{c_1, \ldots, c_r\} \subset V$. Then V is decomposed as

$$V = \bigoplus_{1 \le j \le k \le r} V_{jk} \quad \textit{where} \quad V_{jk} = \left\{ x \in V : L(c_l)x = \frac{\delta_{jl} + \delta_{kl}}{2} x \right\}.$$

Moreover $V_{jj} = \mathbb{R}c_j$ holds, and all V_{jk}'s ($j \ne k$) have the same dimension (see [6, Theorem IV.2.1, Corollary IV.2.6]). We write $\dim V_{jk} = d$. Then $\dim V = n = r + \frac{1}{2} r(r-1)d$ holds.

Let $V_{(l)}^{\mathbb{C}} := \bigoplus_{1 \le j \le k \le l} V_{jk}^{\mathbb{C}}$ ($l = 1, \ldots, r$) and $P_{(l)}$ be the projection on $V_{(l)}^{\mathbb{C}}$. We denote by $\det_{(l)}(x)$ the Jordan determinant on the Jordan algebra $V_{(l)}^{\mathbb{C}}$. We set $\Delta_l(x) := \det_{(l)}(P_{(l)}(x))$ for $x \in V^{\mathbb{C}}$. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, the *generalized power function* on $V^{\mathbb{C}}$ is defined by

$$\Delta_{\mathbf{s}}(x) := \Delta_1^{s_1 - s_2}(x) \Delta_2^{s_2 - s_3}(x) \cdots \Delta_{r-1}^{s_{r-1} - s_r}(x) \Delta_r^{s_r}(x).$$

Using this function, we define the *Gindikin Gamma function* and *Pochhammer symbol* as follows: for $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in (\mathbb{Z}_{\ge 0})^r$,

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-\operatorname{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}} dx, \qquad (\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_\Omega(\mathbf{s} + \mathbf{m})}{\Gamma_\Omega(\mathbf{s})}. \tag{2.3}$$

This integral converges for $\operatorname{Re} s_j > (j-1)\frac{d}{2}$, and both functions are extended meromorphically on $\mathbb{C}^r$ (see [6, Theorem VII.1.1] or [9, Theorem 2.1]). Moreover, we have

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^{r} \left(s_j - (j-1)\frac{d}{2} \right)_{m_j} \qquad \textit{where} \quad (s)_m = s(s+1)\cdots(s+m-1).$$

For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we set $\mathbf{s}^* = (s_r, \ldots, s_1)$. Then we can prove easily

$$(\mathbf{s})_{\mathbf{m}+\mathbf{n}} = (\mathbf{s})_{\mathbf{m}}(\mathbf{s}+\mathbf{m})_{\mathbf{n}}, \qquad (-\mathbf{s}^*)_{\mathbf{m}} = (-1)^{|\mathbf{m}|}\left(\mathbf{s} - \mathbf{m}^* + \frac{n}{r}\right)_{\mathbf{m}^*} \tag{2.4}$$

where $|\mathbf{m}| = m_1 + \cdots + m_r$. Here we identify $\lambda \in \mathbb{C}$ and $(\lambda, \ldots, \lambda) \in \mathbb{C}^r$.

2.4 Polynomials on $V^{\mathbb{C}}$

We set $\mathbb{Z}^r_{++} := \{\mathbf{m} = (m_1, \ldots, m_r) \in (\mathbb{Z}_{\geq 0})^r : m_1 \geq m_2 \geq \cdots m_r \geq 0\}$, and denote the space of holomorphic polynomials on $V^{\mathbb{C}}$ by $\mathcal{P}(V^{\mathbb{C}})$. For $\mathbf{m} \in \mathbb{Z}^r_{++}$, we define $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) := \mathbb{C}\text{-}\operatorname{span}\{\Delta_{\mathbf{m}} \circ l : l \in L^{\mathbb{C}}\}$. Then clearly $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ becomes a $L^{\mathbb{C}}$-module. Moreover, we have

Theorem 2.4 (Hua–Kostant–Schmid, see [6, Theorem XI.2.4])**.**

$$\mathcal{P}(V^{\mathbb{C}}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^r_{++}} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}).$$

These $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$'s are mutually inequivalent, and irreducible as $L^{\mathbb{C}}$-modules.

Since Δ_l vanishes on $\overline{\mathcal{X}_{l-1}}$, all polynomials in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ vanish on $\overline{\mathcal{X}_{l-1}}$ if and only if $m_l \neq 0$.

We write $d_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$, and $\Phi_{\mathbf{m}}(x) := \int_{K_L} \Delta_{\mathbf{m}}(kx)dk$. Then the K_L-fixed subspace in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ is spanned by $\Phi_{\mathbf{m}}$ (see [6, Proposition XI.3.1]).

2.5 Inner products on $\mathcal{P}(V^{\mathbb{C}})$

For $f, g \in \mathcal{P}(V^{\mathbb{C}})$, we define the *Fischer inner product* by

$$\langle f, g\rangle_F := \frac{1}{\pi^n}\int_{V^{\mathbb{C}}} f(w)\overline{g(w)}e^{-(w|w)}dw = f\left(\frac{\partial}{\partial w}\right)\bar{g}(w)\bigg|_{w=0}.$$

Then the reproducing kernel of $\overline{\mathcal{P}(V^{\mathbb{C}})}^F$ (Hilbert completion of $\mathcal{P}(V^{\mathbb{C}})$) is given by $e^{(z|w)}$. We denote by $K^{\mathbf{m}}(z,w) = K^{\mathbf{m}}_w(z)$ the reproducing kernel of $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ with respect to $\langle\cdot,\cdot\rangle_F$. Then clearly,

$$e^{(z|w)} = \sum_{\mathbf{m} \in \mathbb{Z}^r_{++}} K^{\mathbf{m}}(z,w),$$

Also, by [6, Proposition XI.3.3, Propsition XI.4.1.(ii)], we have

$$K^{\mathbf{m}}(gz, w) = K^{\mathbf{m}}(z, g^*w) \qquad for\ \forall g \in Str(V^{\mathbb{C}}),$$

$$K^{\mathbf{m}}_e(z) = \frac{1}{\|\Phi_{\mathbf{m}}\|_F^2}\Phi_{\mathbf{m}}(z) = \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(z)$$

and

$$K^{\mathbf{m}}(x, \bar{x}) = K^{\mathbf{m}}(x^2, e)$$

for $x \in V$, and therefore for any $x \in V^{\mathbb{C}}$ by analytic continuation.

Also, for $\lambda > \frac{2n}{r} - 1$, we define the *weighted Bergman inner product* on D by

$$\langle f, g\rangle_\lambda := \frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)}\int_D f(w)\overline{g(w)}h(w,w)^{\lambda - \frac{2n}{r}}dw.$$

Then, these two inner products are related as follows:

Theorem 2.5 (Faraut–Korányi, see [6, Theorem XIII.2.7])**.** *If $f, g \in \mathcal{P}(V^{\mathbb{C}})$ are decomposed as $f = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} f_{\mathbf{m}}$, $g = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} g_{\mathbf{m}}$ ($f_{\mathbf{m}}, g_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$), then*

$$\langle f, g\rangle_\lambda = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_F. \tag{2.5}$$

Although the left hand side is only defined for $\lambda > \frac{2n}{r} - 1$, the right hand side extends meromorphically for $\lambda \in \mathbb{C}$. Therefore we can redefine $\langle\cdot,\cdot\rangle$ with this formula for any $\lambda \in \mathbb{C}$ by restricting the domain. For $\lambda \in \mathbb{C}$ we set

$$\begin{aligned}\operatorname{rank}\lambda &:= \max\left\{l \in \{0,1,\dots,r\} : (\lambda)_{\mathbf{m}} \neq 0 \ \mathit{for}\ \forall\mathbf{m} \in \mathbb{Z}^r_{++} \cap \{m_{l+1} = 0\}\right\}\\ &= \begin{cases} l & \mathit{if}\ \lambda \in \left(l\frac{d}{2} + \mathbb{Z}_{\le 0}\right) \setminus \bigcup_{j=0}^{l-1}\left(j\frac{d}{2} + \mathbb{Z}_{\le 0}\right) \quad (l = 0, 1, \dots, r-1),\\ r & \mathit{if}\ \lambda \notin \bigcup_{j=0}^{r-1}\left(j\frac{d}{2} + \mathbb{Z}_{\le 0}\right). \end{cases}\end{aligned} \tag{2.6}$$

For example, if $d = 2$, *i.e.*, $V = \mathrm{Herm}(r, \mathbb{C})$, then

$$\operatorname{rank}\lambda = \begin{cases} 0 & (\lambda \in \mathbb{Z}_{\le 0}),\\ l & (\lambda = l,\ l = 1,\dots,r-1),\\ r & (\lambda \notin r - 1 + \mathbb{Z}_{\le 0}). \end{cases}$$

Then $\langle\cdot,\cdot\rangle_\lambda$ defines a sesquilinear form on $\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++},\ m_{\operatorname{rank}\lambda+1}=0} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$. This form $\langle\cdot,\cdot\rangle_\lambda$ is positive definite if and only if

$$\lambda \in \mathcal{W} := \left\{0, \frac{d}{2}, \dots, (r-1)\frac{d}{2}\right\} \cup \left((r-1)\frac{d}{2}, \infty\right). \tag{2.7}$$

This set $\mathcal{W}$ is called the *(Berezin–)Wallach set* (see [17] or [3]).

2.6 Invariant differential operators

For $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{\ge 0}$, we define the differential operators

$$D^{(k)}(\lambda) := \Delta(x)^{\frac{n}{r}-\lambda} \Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda - \frac{n}{r} + k}$$

where $\Delta\left(\frac{\partial}{\partial x}\right)$ is the differential operator characterized by $\Delta\left(\frac{\partial}{\partial x}\right) e^{(x|y)} = \Delta(y) e^{(x|y)}$. Then these operators commute with the $L^{\mathbb{C}}$-action (*i.e.*, $D^{(k)}(\lambda)(f \circ l) = (D^{(k)}(\lambda) f) \circ l$ for $f \in \mathcal{P}(V^{\mathbb{C}})$ and $l \in L^{\mathbb{C}}$). Moreover, we have

Proposition 2.6.

$$D^{(k)}(\lambda) e^{(x|y)} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k} (-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}} (\lambda + \mathbf{m})_{k-\mathbf{m}} K^{\mathbf{m}}(x, y) e^{(x|y)}$$

and if $(\lambda)_{\mathbf{m}} \neq 0$ for any $\mathbf{m} \in \mathbb{Z}^r_{++}$, $|\mathbf{m}| \le k$,

$$D^{(k)}(\lambda) e^{(x|y)} = (\lambda)_k \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k} \frac{(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x, y) e^{(x|y)}.$$

Proof. We follow the proof of [6, Proposition XIV.1.5]. For $x \in \Omega$ and $\lambda < -k+1$,

$$D^{(k)}(\lambda)e^{(x|e)} = \Delta(x)^{\frac{n}{r}-\lambda}\Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda-\frac{n}{r}+k}e^{(x|e)}$$

$$=\Delta(x)^{\frac{n}{r}-\lambda}\Delta\left(\frac{\partial}{\partial x}\right)^k \frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\int_\Omega e^{(x|e-y)}\Delta(x)^{-\lambda+\frac{n}{r}-k}\Delta(y)^{-\frac{n}{r}}dy$$

$$=\Delta(x)^{\frac{n}{r}-\lambda}\frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\int_\Omega e^{(x|e-y)}\Delta(e-y)^k\Delta(x)^{-\lambda+\frac{n}{r}-k}\Delta(y)^{-\frac{n}{r}}dy$$

$$=\Delta(x)^{\frac{n}{r}-\lambda}\frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k} d_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\int_\Omega e^{(x|e-y)}\Phi_{\mathbf{m}-\lambda+\frac{n}{r}-k}(y)\Delta(y)^{-\frac{n}{r}}dy$$

$$=\Delta(x)^{\frac{n}{r}-\lambda}\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k} d_{\mathbf{m}}\frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{\Gamma_\Omega\left(\mathbf{m}-\lambda+\frac{n}{r}-k\right)}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)}\Phi_{-\mathbf{m}^*+\lambda-\frac{n}{r}+k}(x)e^{(x|e)}$$

$$=\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k}\frac{d_{\mathbf{m}}(-k)_{\mathbf{m}}\left(-\lambda+\frac{n}{r}-k\right)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{k-\mathbf{m}^*}(x)e^{(x|e)}$$

$$=\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k}\frac{d_{k-\mathbf{m}^*}(-k)_{k-\mathbf{m}^*}\left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}}\Phi_{\mathbf{m}}(x)e^{(x|e)}.$$

Here we used [6, Proposition VII.1.2], [6, Corollary XII.1.3], and [6, Proposition VII.1.2, VII.1.5] at the 2nd, 4th, and 5th equalities respectively. Now, $d_{\mathbf{m}} = d_{k-\mathbf{m}^*}$ holds since $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) \to \mathcal{P}_{k-\mathbf{m}^*}(V^{\mathbb{C}})$, $p \mapsto \Delta(x)^k p(x^{-1})$ is an isomorphism. Also, by (2.4),

$$\frac{(-k)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|\mathbf{m}|}\left(\frac{n}{r}+\mathbf{m}\right)_{k-\mathbf{m}}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|\mathbf{m}|}\left(\frac{n}{r}\right)_k}{\left(\frac{n}{r}\right)_{\mathbf{m}}\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}},$$

$$\left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*} = (-1)^{|\mathbf{m}|}(\lambda+\mathbf{m})_{k-\mathbf{m}}.$$

Therefore,

$$D^{(k)}(\lambda)e^{(x|e)} = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k}(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x)e^{(x|e)}.$$

By the $L^{\mathbb{C}}$-invariance of $D^{(k)}(\lambda)$, for $y \in \Omega$,

$$D^{(k)}(\lambda)e^{(x|y)} = D^{(k)}(\lambda)e^{(P(y^{\frac{1}{2}})x|e)}$$

$$=\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k}(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\Phi_{\mathbf{m}}((P(y^{\frac{1}{2}})x)e^{(P(y^{\frac{1}{2}})x|e)}$$

$$=\sum_{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\le k}(-1)^{|\mathbf{m}|}(-k)_{\mathbf{m}}(\lambda+\mathbf{m})_{k-\mathbf{m}}K^{\mathbf{m}}(x,y)e^{(x|y)}.$$

This holds for any $x, y \in V^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ by analytic continuation. The second equality follows from

$$(\lambda+\mathbf{m})_{k-\mathbf{m}} = \frac{(\lambda)_k}{(\lambda)_{\mathbf{m}}}.$$

$\square$

Using these differential operators, we can calculate $\langle f,g\rangle_\lambda$ for $\lambda\in\mathbb{C}$: for $\operatorname{Re}\lambda+k>\frac{2n}{r}-1$ and $f,g\in\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++},\ m_{\operatorname{rank}\lambda+1}=0}\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$,

$$\langle f,g\rangle_\lambda=\begin{cases}\dfrac{c_{\lambda+k}}{(\lambda)_k}\displaystyle\int_D(D^{(k)}(\lambda)f)(w)\overline{g(w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw & (\operatorname{rank}\lambda=r)\\ \displaystyle\lim_{\mu\to\lambda}\frac{c_{\mu+k}}{(\mu)_k}\int_D(D^{(k)}(\mu)f)(w)\overline{g(w)}h(w,w)^{\mu+k-\frac{2n}{r}}dw & (\operatorname{rank}\lambda<r)\end{cases}\tag{2.8}$$

where $c_\lambda=\frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda-\frac{n}{r}\right)}$ (see [6, Proposition XIV.2.2, Proposition XIV.2.5]). We can prove easily that this equality holds not only for polynomials, but also for holomorphic functions $f,g\in\mathcal{O}(D)$ with $D^{(k)}(\lambda)f$ and g bounded on $\overline{D}$.

3 Proof for main theorem

For $\lambda\in\mathbb{C}$ with $\operatorname{rank}\lambda=r$, we define the I and J-Bessel functions by

$$\mathcal{I}_\lambda(x):=\sum_{\mathbf{m}\in Z^r_{++}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{1}{(\lambda)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x),$$
$$\mathcal{J}_\lambda(x):=\sum_{\mathbf{m}\in Z^r_{++}}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x)=\mathcal{I}_\lambda(-x).$$

If $\operatorname{rank}\lambda<r$, then $(\lambda)_{\mathbf{m}}=0$ for some $\mathbf{m}$, so we cannot define these functions on entire $V^{\mathbb{C}}$. However, if $x\in\overline{\mathcal{X}_l}$, $\Phi_{\mathbf{m}}(x)=0$ for $m_{l+1}\neq 0$, and therefore for any $\lambda\in\mathbb{C}$ we can define I and J-Bessel functions for $x\in\overline{\mathcal{X}_{\operatorname{rank}\lambda}}$ (see (2.1) and (2.6)) by

$$\mathcal{I}_\lambda(x):=\sum_{\mathbf{m}\in Z^r_{++},\ m_{\operatorname{rank}\lambda+1}=0}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{1}{(\lambda)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x),$$
$$\mathcal{J}_\lambda(x):=\sum_{\mathbf{m}\in Z^r_{++},\ m_{\operatorname{rank}\lambda+1}=0}\frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}}\frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}}\Phi_{\mathbf{m}}(x)=\mathcal{I}_\lambda(-x).$$

Now we are ready to state the main theorem.

Theorem 3.1. *For $\lambda\in\mathbb{C}$, $x\in\overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, take $k\in\mathbb{Z}_{\geq 0}$ such that $\operatorname{Re}\lambda+k>\frac{2n}{r}-1$. Then we have the integral expressions*

$$\mathcal{I}_\lambda\left(x^2\right)=c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-x,w)e^{2(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw,$$
$$\mathcal{J}_\lambda\left(x^2\right)=c_{\lambda+k}\int_D {}_1F_1(-k,\lambda;-ix,w)e^{2i(x|\operatorname{Re}w)}h(w,w)^{\lambda+k-\frac{2n}{r}}dw.$$

where

$$c_\lambda=\frac{1}{\pi^n}\frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda-\frac{n}{r}\right)},\quad {}_1F_1(-k,\lambda;x,w)=\sum_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++},\ |\mathbf{m}|\leq k,\\ m_{\operatorname{rank}\lambda+1}=0}}\frac{(-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}}K^{\mathbf{m}}(x,w).$$

Proof. We calculate $\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda$ in two ways. By (2.5),

$$\begin{aligned}\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda &= \left\langle \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} K^{\mathbf{m}}_{\bar{x}}, \sum_{\mathbf{n}\in\mathbb{Z}^r_{++}} K^{\mathbf{n}}_{x}\right\rangle_\lambda = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}} \left\langle K^{\mathbf{m}}_{\bar{x}}, K^{\mathbf{m}}_{x}\right\rangle_F \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x,\bar{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x^2,e) \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^r_{++}} \frac{1}{(\lambda)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x^2) = \mathcal{I}(x^2).\end{aligned}$$

On the other hand, by (2.8) and Proposition 2.6,

$$\begin{aligned}\left\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)}\right\rangle_\lambda &= \lim_{\mu\to\lambda} \frac{c_{\mu+k}}{(\mu)_k} \int_D \left(D^{(k)}(\mu) e^{(w|\bar{x})}\right) \overline{e^{(w|x)}} h(w,w)^{\mu+k-\frac{2n}{r}} dw \\ &= \lim_{\mu\to\lambda} c_{\mu+k} \int_D {}_1F_1(-k,\mu;-x,w) e^{(w|\bar{x})} \overline{e^{(w|x)}} h(w,w)^{\mu+k-\frac{2n}{r}} dw \\ &= c_{\lambda+k} \int_D {}_1F_1(-k,\lambda;-x,w) e^{2(x|\operatorname{Re} w)} h(w,w)^{\lambda+k-\frac{2n}{r}} dw.\end{aligned}$$

The formula for $\mathcal{J}_\lambda\left(x^2\right)$ follows by replacing x by ix. ☐

From this theorem we can easily deduce the following corollary.

Corollary 3.2. *For $\lambda\in\mathbb{C}$, $x\in\overline{\mathcal{X}_{\operatorname{rank}\lambda}}$, if $\operatorname{Re}\lambda + k > \frac{2n}{r} - 1$ for some $k\in\mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k}>0$ such that*

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k}\left(1+|x|_1^k\right) e^{2|\operatorname{Re} x|_1}, \qquad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k}\left(1+|x|_1^k\right) e^{2|\operatorname{Im} x|_1}$$

where $|x|_1$ is the norm defined in Definition 2.1.

Proof. By Proposition 2.2, for $w\in D$, $x\in V^{\mathbb{C}}$,

$$|(\operatorname{Re} x|\operatorname{Re} w)| \leq |\operatorname{Re} x|_1 |\operatorname{Re} w|_\infty \leq |\operatorname{Re} x|_1 \frac{|w|_\infty + |\bar{w}|_\infty}{2} \leq |\operatorname{Re} x|_1.$$

Also, since ${}_1F_1(-k,\lambda;-x,w)$ is a polynomial of degree k with respect to both x and w,

$$|{}_1F_1(-k,\lambda;-x,w)| \leq C'_{\lambda,k}\left(1+|x|_1^k\right)\left(1+|w|_\infty^k\right) \leq 2C'_{\lambda,k}\left(1+|x|_1^k\right).$$

Therefore, by Theorem 3.1,

$$\begin{aligned}|\mathcal{I}_\lambda(x^2)| &\leq |c_{\lambda+k}| \int_D |{}_1F_1(-k,\lambda;-x,w)| e^{2(\operatorname{Re} x|\operatorname{Re} w)} h(w,w)^{\operatorname{Re}\lambda+k-\frac{2n}{r}} dw \\ &\leq 2|c_{\lambda+k}| C'_{\lambda,k}\left(1+|x|_1^k\right) e^{2|\operatorname{Re} x|_1} \int_D h(w,w)^{\operatorname{Re}\lambda+k-\frac{2n}{r}} dw \\ &= C_{\lambda,k}\left(1+|x|_1^k\right) e^{2|\operatorname{Re} x|_1}.\end{aligned}$$

The proof for $\mathcal{J}_\lambda\left(x^2\right)$ is similar. ☐

Remark 3.3. *In [14, Lemma 3.1] Möllers gave another estimate of $\mathcal{J}_\lambda(x)$:*

$$\left|\mathcal{J}_\lambda\left(x^2\right)\right| \leq C\left(1+|x|_2^2\right)^{\frac{r(2n-1)}{4}} e^{2r|x|_2} \qquad for\ \forall\lambda\in\mathcal{W},\ x\in\overline{\mathcal{X}_{\operatorname{rank}\lambda}}\subset V^{\mathbb{C}}.$$

However, our estimate is sharper because our leading term is given by $e^{2|\operatorname{Im}x|_1}$. *Especially in our estimate $\mathcal{J}_\lambda(x)$ is uniformly bounded on V if $\operatorname{Re}\lambda$ is sufficiently large. This difference comes from that of methods of proofs: in [14] the termwise estimate of the Taylor series was used, while in this paper we use the integral formula. However, termwise estimate of Taylor serieses is often insufficient for getting sharp upper bounds. For example, the bound of cosine function is calculated as follows:*

$$|\cos x| = \left|\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m}\right| \leq \sum_{m=0}^{\infty}\frac{1}{(2m)!}|x|^{2m} \leq \sum_{m=0}^{\infty}\frac{1}{m!}|x|^m = e^{|x|}.$$

However, it is well-known that cosine function is bounded unformly on $\mathbb{R}$. So this bound is not sharp.

4 Applications

For $\lambda > \frac{n}{r}-1$, $t\in\mathbb{C}\setminus\pi i\mathbb{Z}$, $\operatorname{Re}t\geq 0$, we define a integral operator on Ω: for a measurable function $\varphi:\Omega\to\mathbb{C}$, we define

$$\tau_\lambda(t)\varphi(x) := \frac{1}{\Gamma_\Omega(\lambda)}\int_\Omega \varphi(y)\frac{e^{-\coth t(\operatorname{tr}x+\operatorname{tr}y)}}{\sinh^{r\lambda}t}\mathcal{I}_\lambda\left(\frac{1}{\sinh^2 t}P(x^{\frac{1}{2}})y\right)\Delta(y)^{\lambda-\frac{n}{r}}dy.$$

Since $\mathcal{I}_\lambda$ is K-invariant, by [6, Lemma XIV.1.2] we can replace $P(x^{\frac{1}{2}})y$ by $P(y^{\frac{1}{2}})x$.

Remark 4.1. *For $\lambda > \frac{2n}{r}-1$, we define the Laplace transform*

$$\mathcal{L}_\lambda : L^2(\Omega,\Delta(x)^{\lambda-\frac{n}{r}}dx) \longrightarrow L^2(V+\sqrt{-1}\Omega,\Delta(\operatorname{Im}z)^{\lambda-\frac{2n}{r}}dz)\cap\mathcal{O}(V+\sqrt{-1}\Omega)$$

by

$$\mathcal{L}_\lambda\varphi(z) := \frac{2^n}{\Gamma_\Omega(\lambda)}\int_\Omega e^{i(z|x)}\varphi(x)\Delta(2x)^{\lambda-\frac{n}{r}}dx.$$

Then we can prove by the similar method to [6, Theorem XV.4.1] that

$$\begin{aligned}\mathcal{L}_\lambda\tau_\lambda(t)\mathcal{L}_\lambda^{-1}F(z) =&\Delta(-\sin(it)z+\cos(it)e))^{-\lambda}\\ &\times F\left((\cos(it)z+\sin(it)e)(-\sin(it)z+\cos(it)e)^{-1}\right).\end{aligned}$$

Especially, $\tau_\lambda(s)\tau_\lambda(t)=\tau_\lambda(s+t)$ holds for $\lambda > \frac{2n}{r}-1$.

Remark 4.2. *Let E be an Euclidean vector space of dimension N with inner product $(\cdot|\cdot)_E$. Then the Hermite semigroup on $L^2(E)$ is given by*

$$\tilde{\tau}(t)f(\xi) := \frac{1}{(2\pi\sinh t)^{\frac{N}{2}}}\int_E f(\eta)\exp\left(-\frac{1}{2}\coth t(|\xi|_E^2+|\eta|_E^2)+\frac{1}{\sinh t}(\xi|\eta)_E\right)d\eta \tag{4.1}$$

for $f \in L^2(E)$, $t \in \mathbb{C}\setminus \pi i\mathbb{Z}$, $\operatorname{Re} t \geq 0$ (see, e.g., [5, Section 5.2]). From now on we assume there exists an self-adjoint representation $\phi : V \to \operatorname{End}(E)$. We also assume $N > r(r-1)d$. Let $Q : E \to V$ be the quadratic map defined by

$$(\phi(x)\xi|\xi)_E = (x|Q(\xi))_V \qquad \text{for any } x \in V,\ \xi \in E.$$

Let $\Sigma := Q^{-1}(e) \subset E$ be the Stiefel manifold. Then we have

$$\int_\Sigma e^{-i(\xi|\sigma)} d\sigma = \mathcal{J}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2}\right)\right) \tag{4.2}$$

(see [6, Proposition XVI.2.3]). We extend Q to $Q : E^{\mathbb{C}} \to V^{\mathbb{C}}$ bilinearly. Then since $\mathcal{J}_\lambda(x) = \mathcal{I}_\lambda(-x)$ we have

$$\int_\Sigma e^{(\xi|\sigma)} d\sigma = \mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2}\right)\right).$$

If $f \in L^2(E)$ is written as $f(\xi) = F\left(\frac{1}{2}Q(\xi)\right)$ with a function F on V, then (4.1) can be rewritten as

$$\begin{aligned}
\tilde{\tau}(t)f(\xi) &= \frac{1}{(2\pi\sinh t)^{\frac{N}{2}}}\int_E F\left(\frac{1}{2}Q(\eta)\right)\exp\left(-\frac{1}{2}\coth t(|\xi|_E^2 + |\eta|_E^2) + \frac{1}{\sinh t}(\xi|\eta)_E\right)d\eta\\
=&\frac{1}{(\pi\sinh t)^{\frac{N}{2}}}\int_E F(Q(\eta))\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + |\eta|_E^2\right) + \frac{\sqrt{2}}{\sinh t}(\xi|\eta)_E\right)d\eta\\
=&\frac{1}{\Gamma_\Omega(\frac{N}{2r})\sinh^{\frac{N}{2}} t}\int_\Omega\int_\Sigma F(Q(\phi(y^{\frac{1}{2}})\sigma))\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + |\phi(y^{\frac{1}{2}})\sigma|_E^2\right)\right)\\
&\times\exp\left(\frac{\sqrt{2}}{\sinh t}(\xi|\phi(y^{\frac{1}{2}})\sigma)_E\right)\Delta(y)^{\frac{N}{2r}-\frac{n}{r}}d\sigma dy\\
=&\frac{1}{\Gamma_\Omega(\frac{N}{2r})}\int_\Omega\int_\Sigma F(y)\frac{\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + \operatorname{tr} y\right)\right)}{\sinh^{\frac{N}{2}} t}\exp\left(\frac{\sqrt{2}}{\sinh t}(\phi(y^{\frac{1}{2}})\xi|\sigma)_E\right)\Delta(y)^{\frac{N}{2r}-\frac{n}{r}}d\sigma dy\\
=&\frac{1}{\Gamma_\Omega(\frac{N}{2r})}\int_\Omega F(y)\frac{\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + \operatorname{tr} y\right)\right)}{\sinh^{\frac{N}{2}} t}\mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{1}{\sqrt{2}\sinh t}\phi(y^{\frac{1}{2}})\xi\right)\right)\Delta(y)^{\frac{N}{2r}-\frac{n}{2r}}dy\\
=&\frac{1}{\Gamma_\Omega(\frac{N}{2r})}\int_\Omega F(y)\frac{\exp\left(-\coth t\left(\frac{1}{2}\operatorname{tr} Q(\xi) + \operatorname{tr} y\right)\right)}{\sinh^{\frac{N}{2}} t}\mathcal{I}_{\frac{N}{2r}}\left(\frac{1}{2\sinh^2 t}P(y^{\frac{1}{2}})Q(\xi)\right)\Delta(y)^{\frac{N}{2r}-\frac{n}{2r}}dy\\
=&\tau_{\frac{N}{2r}}(t)F\left(\frac{1}{2}Q(\xi)\right)
\end{aligned}$$

where we used [6, Proposition XVI.2.1] at the 3rd equality and [6, Lemma XVI.2.2.(ii)] at the 4th, 6th equalities. Therefore $\tau_{\frac{N}{2r}}(t)$ coincides with the action of the Hermite semigroup on radial functions on E.

Remark 4.3. *For $x \in \overline{\mathcal{X}_1}$ (see (2.1)), $\mathcal{I}_\lambda(x) = \Gamma(\lambda)\tilde{I}_{\lambda-1}(2\sqrt{|x|_2})$ holds (see [14, Example 3.3]), and by analytic continuation the distribution $\frac{1}{\Gamma_\Omega(\lambda)}\Delta(x)^{\lambda-\frac{n}{r}}\mathbf{1}_\Omega dx$ at $\lambda = \frac{d}{2}$ gives the semi-invariant measure on $\overline{\mathcal{X}_1}\cap\overline{\Omega}$ (see [6, Proposition VII.2.3]). Therefore for $V = \mathbb{R}^{1,n-1}$ the action τ_λ at $\lambda = \frac{d}{2}$ coincides with the action of the holomorphic semigroup on the minimal representation of $O(p,2)$ (see [12, Theorem B] or [13, Theorem 5.1.1]).*

We set $K_\lambda(x,y;t) := e^{-\coth t(\operatorname{tr} x + \operatorname{tr} y)}\mathcal{I}_\lambda\left(\sinh^{-2} t P(x^{\frac{1}{2}})y\right)$, the kernel function of $\tau_\lambda(t)$. Then we can deduce from Theorem 3.2 that

Theorem 4.4. *Take $k \in \mathbb{Z}_{\geq 0}$ such that $\lambda + k > \frac{2n}{r} - 1$. Then if $t = u + iv$, $u, v \in \mathbb{R}$, $u \geq 0$,*

$$|K_\lambda(x,y;t)| \leq C_\lambda \left(1 + (\operatorname{tr} x \operatorname{tr} y)^{\frac{k}{2}}\right) \exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr} x + \operatorname{tr} y)\right).$$

Especially, if $u = \operatorname{Re} t > 0$ then $\tau_\lambda(t)$ maps functions of polynomial growth to functions of exponential decay. In order to prove this theorem, we prepare the following lemma.

Lemma 4.5. *(1) For $x \in \Omega$ the directional derivative of $x \mapsto \sqrt{x}$ is*

$$D_u\sqrt{x} = \frac{1}{2}L\left(\sqrt{x}\right)^{-1}u.$$

(2) For $x, y \in V$ if $[L(x), L(y)] = 0$, then there exists a Jordan frame $\{c_1, \ldots, c_r\}$ such that $x, y \in \mathbb{R}\text{-}\operatorname{span}\{c_1, \ldots, c_r\}$.

(3) For $x, y \in \Omega$, $\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y} \leq \sqrt{\operatorname{tr} x \operatorname{tr} y} \leq \dfrac{\operatorname{tr} x + \operatorname{tr} y}{2}$.

Proof. (1) $u = D_u x = D_u\left(\sqrt{x}\right)^2 = 2\sqrt{x}D_u\sqrt{x} = 2L\left(\sqrt{x}\right)D_u\sqrt{x}$ and then $D_u\sqrt{x} = \frac{1}{2}L\left(\sqrt{x}\right)^{-1}u$ follows.
(2) See [6, Lemma X.2.2].
(3) The second inequality is clear. For the first inequality, we take $k_0 \in K$ such that $\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})ky}$ $(k \in K_L)$ attains its maximum at $k = k_0$. We put $k_0 y =: y_0$. Then for any $D \in \mathfrak{k}_\mathfrak{l} = Lie(K_L)$,

$$\begin{aligned}
0 = &\left.\frac{d}{dt}\right|_{t=0} \operatorname{tr}\sqrt{P(x^{\frac{1}{2}})e^{tD}y_0} = \frac{1}{2}\operatorname{tr}\left(L\left(\sqrt{P(x^{\frac{1}{2}})y_0}\right)^{-1}P(x^{\frac{1}{2}})Dy_0\right)\\
=&\frac{1}{2}\left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\middle| P(x^{\frac{1}{2}})Dy_0\right) = \frac{1}{2}\left(P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\middle| Dy_0\right).
\end{aligned}$$

We put $P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} =: z$. If $D = [L(u), L(v)]$ $(u, v \in V)$, then

$$\begin{aligned}
0 =&(z|[L(u), L(v)]y_0) = (z|u(vy_0)) - (z|v(uy_0)) = (zu|vy_0) - (zv|uy_0)\\
=&(y_0(zu)|v) - (v|(uy_0)z) = ([L(y_0), L(z)]u|v).
\end{aligned}$$

Since $(\cdot|\cdot)$ is non-degenerate, $[L(y_0), L(z)] = 0$. Also,

$$\begin{aligned}
P(z)y_0 &= P\left(P(x^{\frac{1}{2}})\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\right)y_0\\
&= P(x^{\frac{1}{2}})P\left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1}\right)P(x^{\frac{1}{2}})y_0 = P(x^{\frac{1}{2}})e = x.
\end{aligned}$$

So especially $[L(x), L(y_0)] = 0$. Let $x = \sum_{j=1}^r t_j c_j$, $y = \sum_{j=1}^r s_j d_j$ $(t_j, s_j > 0$, and $\{c_j\}_{j=1}^r, \{d_j\}_{j=1}^r$ are Jordan frames). Then,

$$\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y} \leq \max_{k\in K_L} \operatorname{tr}\sqrt{P(x^{\frac{1}{2}})ky} = \max_{\sigma\in\mathfrak{S}_n} \operatorname{tr}\sqrt{P\left(\sum_{j=1}^r t_{\sigma(j)}^{\frac{1}{2}} c_j\right)\sum_{j=1}^r s_j c_j}$$

$$= \max_{\sigma\in\mathfrak{S}_n} \sum_{j=1}^r \sqrt{t_\sigma(j) s_j} \leq \sqrt{\left(\sum_{j=1}^r t_j\right)\left(\sum_{j=1}^r s_j\right)} = \sqrt{\operatorname{tr} x \operatorname{tr} y}$$

and the proof is completed. □

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. By Corollary 3.2,

$$|K_\lambda(x,y;t)| \leq C'_\lambda e^{-\operatorname{Re}\coth t(\operatorname{tr} x + \operatorname{tr} y)}\left(1 + \left|\frac{1}{\sinh t}\sqrt{P(x^{\frac{1}{2}})y}\right|_1^k\right) e^{2\left|\operatorname{Re}\frac{1}{\sinh t}\sqrt{P(x^{\frac{1}{2}})y}\right|_1}$$

$$= C'_\lambda e^{-\operatorname{Re}\coth t(\operatorname{tr} x + \operatorname{tr} y)}\left(1 + \frac{1}{|\sinh t|^k}\operatorname{tr}\left(\sqrt{P(x^{\frac{1}{2}})y}\right)^k\right) e^{2\operatorname{Re}\frac{1}{\sinh t}\operatorname{tr}\left(\sqrt{P(x^{\frac{1}{2}})y}\right)}$$

$$\leq C_\lambda \exp\left(-\frac{\cosh u \sinh u}{\cosh^2 u - \cos^2 v}(\operatorname{tr} x + \operatorname{tr} y)\right)\left(1 + \sqrt{\operatorname{tr} x \operatorname{tr} y}^k\right)$$

$$\times \exp\left(\frac{\sinh u |\cos v|}{\cosh^2 u - \cos^2 v}(\operatorname{tr} x + \operatorname{tr} y)\right)$$

$$= C_\lambda\left(1 + (\operatorname{tr} x \operatorname{tr} y)^{\frac{k}{2}}\right)\exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr} x + \operatorname{tr} y)\right)$$

and this completes the proof. □

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References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.

[2] I. Barchini, M. Spanski, and R. Zierau, *Positivity of zeta distributions and small unitary representations*, The ubiquitous heat kernel, 1–46, Contemp. Math., 398, Amer. Math. Soc., Providence, RI, 2006.

[3] F. A. Berezin, *Quantization in complex symmetric spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 363–402, 472.

[4] P. H. Dib, *Fonctions de Bessel sur une algèbre de Jordan*, J. Math. Pures Appl. (9) 69 (1990), no. 4, 403–448.

[5] G. B. Folland, *Harmonic analysis in phase space*, Ann. of Math. Studies 122, Princeton, NJ, Princeton University Press, 1989.

[6] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

[7] J. Faraut, S. Kaneyuki, A. Korányi, Q.k. Lu, and G. Roos, *Analysis and geometry on complex homogeneous domains*, Progress in Mathematics, 185, Birkhauser Boston, Inc., Boston, MA, 2000.

[8] J. Faraut and G. Travaglini, *Bessel functions associated with representations of formally real Jordan algebras*, J. Funct. Anal. **71** (1987), no. 1, 123–141.

[9] S. G. Gindikin, *Analysis in homogeneous domains*, Uspehi Mat. Nauk **19** 1964 no. 4 (118), 3–92.

[10] C. S. Herz, *Bessel functions of matrix argument*, Ann. of Math. (2) **61**, (1955), 474–523.

[11] J. Hilgert, T. Kobayashi, J. Möllers, and B. Ørsted, *Fock model and Segal-Bargmann transform for minimal representations of Hermitian Lie groups*, J. Funct. Anal. 263 (2012), no. 11, 3492–3563.

[12] T. Kobayashi and G. Mano. *Integral formulas for the minimal representations for O(p,2)*, Acta Appl. Math., **86**, 2005, no. 1–2, 103–113.

[13] T. Kobayashi and G. Mano, *The inversion formula and holomorphic extension of the minimal representation of the conformal group*, Harmonic analysis, group representations, automorphic forms and invariant theory: In honor of Roger Howe, World Scientific, 2007, 159–223.

[14] J. Möllers, *A geometric quantization of the Kostant-Sekiguchi correspondence for scalar type unitary highest weight representations*, preprint, arXiv:1205.5171.

[15] R. J. Muirhead, *Systems of partial differential equations for hypergeometric functions of matrix argument*, Ann. Math. Statist. **41**, 1970, 991–1001.

[16] R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 27 Springer-Verlag, Berlin-Gottingen-Heidelberg, 1960.

[17] N. R. Wallach, *The analytic continuation of the discrete series I, II*, Trans. Amer. Math. Soc. **251** (1979), 1–17, 19–37.